Journal of Quantitative Economics, Vol. 8 No.1, January 2010

A NOTE ON THE BASIC LEMMA OF THE LINEAR IDENTIFICATION PROBLEM

OSKAR MARIA BAKSALARY[1]
GÖTZ TRENKLER[2]

Abstract

In this note, the basic lemma of the linear identification problem is revisited. By utilizing a joint decomposition of orthogonal projectors as partitioned matrices, a new proof of the lemma is proposed. From the algebraic point of view, the present proof might be the simplest from among all available in the literature till now.

Keywords: Orthogonal projector, Partitioned matrix, Matrix rank, Linear simultaneous equation system, Linear restrictions.

JEL Classification: C13, C30

1. Introduction

A crucial role in the linear identification problem is played by the lemma leading to criteria for identification of a linear simultaneous equation system subject to linear restrictions. In the present note, this, so called, basic lemma of the linear identification problem, is revisited. By utilizing a joint decomposition of orthogonal projectors as partitioned matrices, a new proof of it is proposed. From the algebraic point of view, the present proof might be the simplest from among all available in the literature till now.

The considerations below concern matrices of real entries. The symbols R', R(K), and rk(K) stand for the transpose, column space (range), and rank of matrix K, respectively. Moreover, I_N means the identity matrix of order N, and (. : .) denotes a columnwise partitioned matrix.

An essential role in the present note is played by orthogonal projectors, i.e., symmetric idempotent matrices. It is known that a given matrix is an orthogonal projector if and only if it is expressible as $KK^{\dagger}$ for some matrix K, where $K^{\dagger}$ is the Moore–Penrose inverse of K, i.e., the unique solution to the equations

[1] Faculty of Physics, Adam Mickiewicz University, ul. Umultowska 85, PL 61-614 Poznań, Poland

Corresponding author. Address for correspondence: Department of Statistics, Dortmund University of Technology, Vogelpothsweg 87, D-44221 Dortmund, Germany. Email: baxx@amu.edu.pl (O.M. Baksalary).

Oskar Maria Baksalary would like to express his sincere thanks to the Alexander von Himboldt Foundation for its financial support.

[2] Department of Statistics, Dortmund University of Technology, Vogelpothsweg 87, D-44221 Dortmund, Germany. Email: trenkler@statistik.uni.dortmund.de (G. Trenkler).

$KK^{\dagger}K = K, K^{\dagger}KK^{\dagger} = K^{\dagger}, (KK^{\dagger})' = KK^{\dagger}, (K^{\dagger}K)' = K^{\dagger}K.$

Then $P_K = KK^{\dagger}$ is the orthogonal projector onto R(K).

The lemma below will be helpful in establishing subsequent results. It should be stressed, however, that the two rank characterizations provided therein prove to be useful in various considerations within matrix theory, statistics, and econometrics.

LEMMA. Let F and G be matrices with the same number of rows, and let $P = P_F$ and $Q = P_G$ Then:

(i) rk($F'G$) = rk(PQ),

(ii) rk(F : G) = rk(P + Q).

Proof. On account of the properties of rank and the Moore–Penrose inverse, we have

$$\begin{aligned} rk(F'G) &= rk(F'GG^{\dagger}G) \le rk(F'Q) \le rk(F'G) = rk(G'F) \\ &= rk(G'FF^{\dagger}F) \le rk(G'P) \le rk(G'F) = rk(F'G) \end{aligned}$$

Thus, we have shown that rk($F'G$) = rk($F'Q$) = rk(PG). Replacing G with Q gives rk($F'Q$) = rk(PQ), establishing point (i) of the lemma.

The proof of point (ii) is obtained from the following relationships on column spaces

$$\begin{aligned} R(F:G) &= R[(F:G)(F:G)'] = R(FF' + GG') = R(FF') + R(GG') \\ &= R(F) + R(G) = R(P) + R(Q) = R(P + Q), \end{aligned}$$

where the use was made of the known fact that range of two nonnegative definite matrices is additive.

2. Main Results

Adopting the notation of Johnston (1984, Sec. 11-2), the structural form of a simultaneous equation system may be written as

$$Az_t = (B:\Gamma)\begin{pmatrix} y_t \\ x_t \end{pmatrix} = u_t, \qquad \dots (1)$$

where A = (B : Γ) is the G x (G + K) matrix of the structural coefficients and $z_t = (y_t' : x_t')'$ is the (G +K) x1 vector of observations on all variables at time t. As a common assumption, it is required that the G x G matrix B is nonsingular. Setting

$$\Pi = -B^{-1}\Gamma \qquad \dots (2)$$

We have $B\Pi + \Gamma = 0$, or, equivalently,

$$AW=0. \qquad \dots (3)$$

where

$$W = \begin{pmatrix} \Pi \\ I_K \end{pmatrix} \qquad \dots (4)$$

is a (G + K) x K matrix.

If α_1' denotes the upper row of A, the first structural equation can be expressed as

$$\alpha_1' z_t = u_{1t},$$

with u_{1t} denoting the first entry of the disturbance vector u_t. Suppose we have a priori restrictions on α_1 in the form

$$\alpha_1' \Phi = 0, \qquad \ldots (5)$$

where Φ is a known (G + K) x R matrix, whose number of columns R corresponds to the number of a priori restrictions on the first equation. Relationship (3) implies that

$$\alpha_1' W = 0 \qquad \ldots (6)$$

whence, by combining (5) and (6), we arrive at

$$\alpha_1'(W : \Phi) = 0,$$

being a set of K + R equations with G + K unknowns. Having normalized the first equation by fixing one of its coefficients at unity, a necessary and sufficient condition to get a unique solution for the (G+K) x 1 vector α_1 is

$$\mathrm{rk}(W : \Phi) = G + K - 1 \qquad \ldots (7)$$

As pointed out by Farebrother (1971), criteria (7) is not convenient in application as it requires the construction of the matrix Π defined in (2). This is not the case with the result proposed by Fisher (1966, Lemma 2.6.1), which expresses rank of (W: Φ) in terms of rank of $A\Phi$. This result, now called the basic lemma of the linear identification problem, is recalled below. It is accompanied by an original proof utilizing the notion of orthogonal projectors, which seems to be relatively simple in comparison with other proofs available in the literature; for alternative proofs see e.g., Farebrother (1971) or Schmidt (1976, p. 136).

THEOREM 1. Let A, W, and Φ be matrices defined in (1), (4), and (5), respectively. Then

$$\mathrm{rk}(W : \Phi) = \mathrm{rk}(A\Phi) + K.$$

Proof. Consider (G + K) x (G + K) matrix

$$(A' : W) = \begin{pmatrix} B' & \Pi \\ \Gamma' & I_K \end{pmatrix}. \qquad \ldots (8)$$

In view of (2), by a well-known formula for the determinant of a partitioned matrix (see e.g., Graybill (1983, Theorem 8.2.1)), from (8) we obtain

$$\det(A' : W) = \det(B')\det[I_K - \Gamma'(B')^{-1}\Pi] = \det(B')\det(I_K + \Pi'\Pi).$$

Since B is nonsingular and $I_K + \Pi'\Pi$ is positive definite, $(A' : W)$ is nonsingular as well. Let P and Q_1 be the orthogonal projectors onto R(W) and $R(A')$, respectively, i.e., P = P_W and $Q_1 = P_{A'}$. From (4) it is seen that rk(W) = K , and, thus, P can be represented in the form

$$P = U\begin{pmatrix} I_K & 0 \\ 0 & 0 \end{pmatrix}U'$$

Where U is a (G+K) x (G+K) orthogonal matrix; see e.g. Trenkler (1994, Theorem 13). Accordingly, we can write

$$P = U\begin{pmatrix} A_1 & B_1 \\ B_1' & D_1 \end{pmatrix}U',$$

With A_1 being K x K matrix. Noting that Lemma entails

$$rk(AW) = rk(Q_1P) \quad \text{and} \quad rk(A' : W) = rk(Q_1 + P)$$

from Baksalary and Trenkler (2008, Lemma 6), we get

$$rk(AW) = rk(A_1) \quad \text{and} \quad rk(A' : W) = rk(D_1) + K \qquad \ldots (9)$$

Combining the first equality in (9) with (3) leads to $rk(A_1) = 0$, whereas, in the light of the nonsingularity of $(A' : W)$, the second equality in (9) ensures that D_1 is nonsingular. On account of Baksalary and Trenkler (2008, Lemma 1), the first of these observations implies $B_1 = 0$, while the second yields $D_1 = I_G$. In consequence,

$$Q_1 = U\begin{pmatrix} 0 & 0 \\ 0 & I_G \end{pmatrix}U' = I_{G+K} - P$$

To complete the proof, let

$$Q_2 = U\begin{pmatrix} A_2 & B_2 \\ B_2' & D_2 \end{pmatrix}U',$$

With A_2 being the K x K matrix, represent the orthogonal projector onto $R(\Phi)$. Analogous derivations to those leading to relationships (9), give now

$$rk(A\Phi) = rk[(I_{G+K} - P)Q_2] = rk(D_2) \qquad \ldots (10)$$

and

$$rk(W : \Phi) = rk(P + Q_2) = rk(D_2) + K,$$

from where the assertion is clearly seen.

Theorem 1 is supplemented with some observations. Firstly, note that from Theorem 1 and (7) it is seen that conditions

$$rk(W : \Phi) = G + K - 1 \quad \text{and} \quad rk(A\Phi) = G - 1 \qquad \ldots (11)$$

are equivalent; wee Schmidt (1976, p. 134). Secondly, from Baksalary and Trenkler (2008, Lemma 5) it follows that

$$R(W) \cap R(\Phi) = \{0\} \Leftrightarrow R(P) \cap R(Q_2) = \{0\} \Leftrightarrow rk(A_2) = rk(B_2).$$

Combining these characterizations with (10) and relationship $rk(\Phi) = rk(Q_2) = rk(A_2) - rk(B_2) + rk(D_2)$, given in Baksalary and Trenkler (2008, Lemma 1), leads to the theorem below.

THEOREM 2. Let A, W and Φ be matrices defined in (1), (4), and (5), respectively. Then $rk(A\Phi) = rk(\Phi)$ if and only if $R(W) \cap R(\Phi) = \{0\}$

In view of the above, it is clear that when $R(W) \cap R(\Phi) = \{0\}$, then the right hand side condition in (11) can be expressed as $rk(\Phi) = G - 1$ in which case $rk(W : \Phi) = rk(\Phi) + K$.

References

Baksalary, O.M. & G. Trenkler (2008), "An alternative approach to characterize the commutativity of orthogonal projectors", *Discussiones Mathematicae – Probability and Statistics*, 28, 113-137.

Farebrother, R.W. (1971), "A Short Proof of the Basic Lemma of the Linear Identification Problem", *International Economic Review*, 12, 515-516.

Fisher, F.M. (1966), *The Identification Problem in Econometrics*, New York, McGraw-Hill.

Graybill, F.A. (1983), *Matrices with Applications to Statistics*, Belmont: Wadsworth Publishing Company.

Johnston, J. (1984), *Econometric Methods*, New York, McGraw-Hill.

Schmidt, P. (1976), *Econometrics*, New York, Marcel Dekker.

Trenkler, G. (1994), "Characterizations of Oblique and Orthogonal Projectors", in T. Calinski, R. Kala (eds.) *Proceedings of the International Conference on Linear Statistical Inference LINSTAT'93*, Dordrecht, Kluwer, pp. 255-270.

♦♦♦